Forecasting Life Expectancy in an International Context

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1 Introduction

Many factors influencing mortality are not limited to their country of discovery - both germs and medical advances can transcend borders. How can we consider the impact of common factors on the future mortality of different countries? We suggest using a country-aggregated measure of mortality, the best-practice line - the average length of life in the best-practice population - to obtain better forecasts of life expectancy at the national level.

Over the past 160 years, female best-practice life expectancy has been extraordinarily linear, with a steadily increase by a quarter of a year per year. This phenomenon was first observed by Oeppen and Vaupel (2002) who acknowledge in their findings the "cogent evidence" that the expectation of life can rise much further. Indeed, if life expectancy were close to a maximum, then the increase in the record expectation of life should be slowing, and this is not the case.

Oeppen and Vaupel suggested forecasting life expectancy in individual countries by considering the gap in national performance in comparison to the best-practice level. In this direction, attempts have been made by Andreev and Vaupel (2006) who assumed a constant future value of the gap, and by Lee (2006) who assumed that life expectancy tends to increase at some constant rate and, additionally, each year moves of a certain proportion of the amplitude of the gap towards the best-practice line. The first model seems to me too simplistic and the other one not adequate to guarantee future values of life expectancy not exceeding the best-practice line.

We will forecast life expectancy in individual countries combining the two distinct forecasts of the best-practice line and the gap. The best-practice

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line is characterized by a striking linear trend, while the gap seems to have exponentially decreasing trend, since, as Oeppen and Vaupel pointed out, countries lagging behind tend to catch up with the best-practice population. The assumption is further supported by White's (2002) assertion that "globalization may be occurring among rich countries, in practice affecting mortality, as well as everything else. This could bring to converging mortality patterns". To justify such trend for the countries lagging behind the best-practice line, it can be hypothesized that corrective actions should be undertaken by the government if a country begins to fall too far behind. For example in Denmark, committees have been appointed to investigate possible means of reducing dangerous behavior (e.g. smoking and alcohol consumption, both of which can be influenced by education and regulation) and the inadequacy of health investments in the past (Bengtsson, 2006).

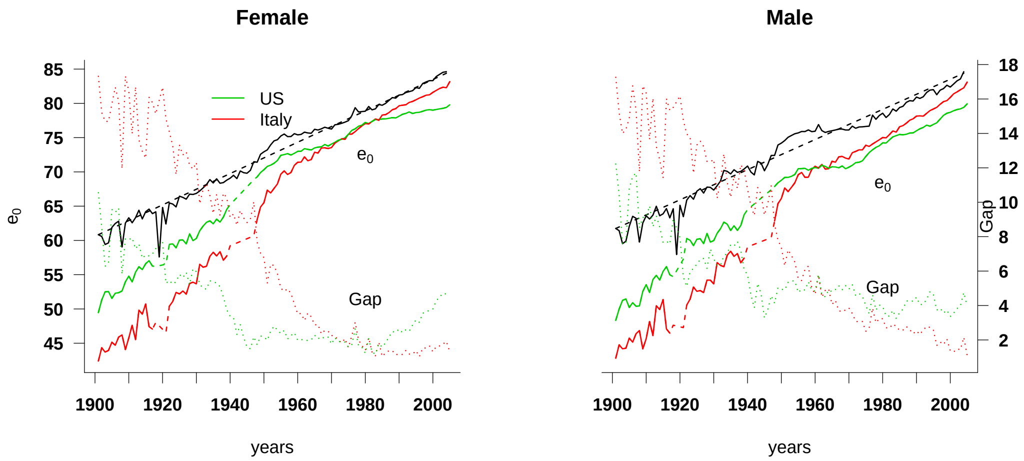
The time series of the best-practice level is modeled and forecast with the classic univariate ARIMA model (Box and Jenkins, 1976). The gap is fitted with two different models derived from the financial economics theory. Uncertainty in the forecast is combined together via Monte Carlo simulation.

2 Data description

The analysis is conducted using data coming from the Human Mortality Database (2008). We look at life expectancy in Italy and the US, on the period with available data for both countries, going from 1900 until 2003. Life expectancies at birth in Italy and the US, together with the best-practice line and the two corresponding gaps are plotted in Figure 1.

The slower increase in life expectancies occurring in the middle of the century, reflect the shift in the primary causes of death. Specifically, the reduction in infant and child mortality observed before 1950 was substituted by the reduction in old mortality starting in the late 1960s. Looking at the relationship between life expectancy and the best-practice line we can observe an initial convergence of the Italian female life expectancy, that set on constant values in the last three decades. Contrary, US female life expectancy got closer and then lagged behind the best-practice line. In 1900, female life expectancy in the US was 49 years and was lagging roughly 10 years behind the world record leader, New Zealand. Later on the US managed to catch up, recording a gap of only two years in 1950 with some oscillation until approximately 1980. Since 1980 the gains in life expectancy started to reduce, and the gap widened. In the year 2000, US female life expectancy was 79.7 years, again about five years lower than the record life expectancy in Japan. The plot of male life expectancies shows a convergent trend towards

Figure 1: Life expectancies at birth calculated using Italian and US data, together with the best-practice line (solid black), the fitted best-practice line (dashed black) and the two corresponding gaps.



the best-practice line until 1950, and almost constant value onwards.

3 Methods

Using the advantage of the striking linear trend observed in the best-practice line, and assuming the persistence of the observed past trends, we extrapolate the stochastic process using the classic ARIMA models. The model selection strategy developed by Box and Jenkins (1976) helps to select an adequate model following three phases: the model identification; the model selection and the diagnostic checking of the model adequacy. Selected the adequate model, forecast of the future values are performed.

Thinks are a bit more complicated when we want model the gap. Considering the best-practice line as the upper bound for the country specific life expectancy, the lower bound for the gap is equal to zero. Therefore, the gap should be modelled in such a way that it never exceeds the value of zero. If this condition is not met, we will obtain future values of life expectancy in the individual countries higher than the best-practice line. Namely we obtain a pattern of the best-practice line diverging from what we expected. We decide to work with the logarithm transformation of the gaps, guaranteeing the positiveness of future values.

We suggest here to model the stochastic future values of the gaps with two alternative models derived from financial economics theory: the geometric Brownian motion and the geometric mean reversion process.

3.1 Geometric Brownian motion

The geometric Brownian motion is a continuous-time stochastic process in which the logarithm of the randomly varying quantity follows a Brownian motion. The model is usually used in financial economics to describe the behavior of stock prices. It assumes that the mean and variability of stock prices are proportional to the value of the prices itself. A stochastic process X following a geometric Brownian motion satisfies the following stochastic differential equation:

$$dX = \mu X dt + \sigma X dz \tag{1}$$

where z is a Wiener process, and the constant μ and σ are the percentage drift and volatility of X. The first term on the right hand side of the equation is the expected variation of X in the time interval dt given the drift term, and the second term represent the variation of X in the time interval dt due to the random component.

To obtain positive values of the variable X, we work with the logarithmic transformation of the random variable X. Applying then Itô's lemma to approximate the variation of the new function $F = \ln(X)$ to changes occurring in the variable X and time t, we have:

$$dF = \left(\mu - \frac{\sigma^2}{2}\right) dt + \sigma\, dz = \mu' dt + \sigma\, dz \tag{2}$$

The random variable F_t is normally distributed with mean $F_0 + \left(\mu - \frac{\sigma^2}{2}\right) t$ and variance $\sigma^2 t$. Integrating both members of equation (2) we obtain:

$$F_t - F_0 = \int_0^t dF = \mu' \int_0^t ds + \sigma \int_0^t dz = \mu' t + \sigma(z_t - z_0) \tag{3}$$

substituting $F = \ln(X)$ we obtain the following solution:

$$X_t = X_0 \exp\left\{ \left(\mu - \frac{\sigma^2}{2}\right) t + \sigma\left(z_t - z_0\right) \right\} \tag{4}$$

If the logarithm of a variable is normally distributed, the variable itself is log-normally distributed. Hence X_t is log-normally distributed with the mean and variance, obtained through the moment-generating function of a normal distribution, equal to:

$$E(X_t) = X_0 e^{\mu t} \qquad \text{and} \qquad Var(X_t) = X_0^2 e^{2\mu t} \left(e^{\sigma^2 t} - 1 \right)$$

Future values of the stochastic variable X are obtained through simulations of the paths of the stochastic variable $F = \ln(X)$, to whom is then applied the exponential function. The values of the logarithm of X in t are obtained using the following equation:

$$\ln(X_t) = \ln(X_0) + \left(\mu - \frac{\sigma^2}{2} \right) t + \sigma \sqrt{t}\, \varepsilon \tag{5}$$

where the ε are drawn from a standard normal distributions.

3.2 Geometric mean reversion model

The geometric mean reversion model, also called the mean-reversion Ornstein-Uhlenbeck model (Uhlenbeck and Ornstein, 1930), is characterized by a non-explosive behavior that tends to fluctuate around the reversion level. The process is further characterized by a limited long-term variance. It is used in financial economics to describe the behavior of the prices of commodities, where the demand and supply forces act when the prices are far from a "more reasonable" equilibrium level. The formula is given by the following stochastic process:

$$dX = \eta X \left[m - log(X) \right] dt + \sigma X dz \tag{6}$$

where m is the long-run equilibrium level of the logarithm of X, η is the speed of reversion, z is a Wiener process, and σ is the volatility of X. The difference between the geometric mean-reversion process and the geometric Brownian motion is in the drift term: the drift is positive if the current level of X is lower than the equilibrium level m and viceversa. In other words, the equilibrium level attracts the process in its direction and the more distant are the prices from the equilibrium level, the higher is the tendency to revert back to the level m.

Also here, to obtain positive values of the variable X, we work with the logarithmic transformation of the random variable X. Applying then

Itô's lemma to approximate the variation of the new function $F = \ln(X)$ to changes occurring in the variable X and time t, we have:

$$dF = \eta \left[\left(m - \frac{\sigma^2}{2\eta} \right) - F \right] dt + \sigma dz = \eta(m' - F)dt + \sigma dz \tag{7}$$

Integrating the above stochastic differential equation we obtain:

$$F_t = F_0 e^{-\eta t} + (1 - e^{-\eta t})m' + \sigma e^{-\eta t} \int_0^t e^{\eta s} dz(s) \tag{8}$$

The random variable F_t is normally distributed with parameters:

$$E(F_t) = F_0 e^{-\eta t} + (1 - e^{-\eta t})\, m' \quad \text{and} \quad Var(F_t) = (1 - e^{-2\eta t}) \cdot \frac{\sigma^2}{2\eta}$$

The mean is a weighted average between the initial level F and the long-run level m', and the variance increases with time, but it also converges to $\sigma^2/(2\eta)$ as the time goes to infinity. If the logarithm of a variable X_t is normally distributed, the variable itself X_t is log-normally distributed. More information concerning the parameters of the log-normal distribution are provided in Oksendal (1995).

Future values of the stochastic variable X are obtained through simulations of the paths of the stochastic variable $F = \ln(X)$, to whom is then applied the exponential function. The values of the logarithm of X in t are obtained using the following discrete equation:

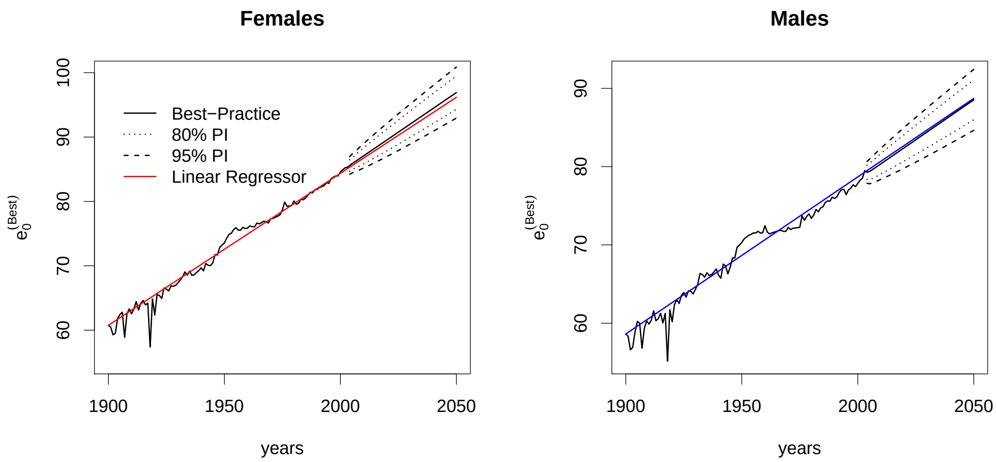
$$\ln(X_t) = m' \left(1 - e^{-\eta \Delta t}\right) + e^{-\eta \Delta t} \ln(X_{t-1}) + \varepsilon_t \tag{9}$$

where $\Delta t = 1$, and ε_t is drawn from a normal distribution with mean 0 and variance $\sigma_\varepsilon^2 = (1 - e^{-2\eta}) \frac{\sigma^2}{2\eta}$.

4 Application

The methods described in Section 3 are applied to Italian and US data on the period going from 1900 until 2003. Generally, to forecast mortality rates or directly life expectancy we would consider data going from 1950 until 2003, assuming that the reduction in mortality observed in the second half of the century at old ages, will persist in the future. Here, instead, we are

Figure 2: Forecast best-practice line, with 80% and 95% prediction intervals.



interested to model the existing relationship between life expectancy in a specific country and the best-practice line. The latter, additionally did not experience any change in the middle of the century. Looking at the data available for the whole century we aim to model the converging behavior between life expectancy in individual countries and the best-practice line.

4.1 Forecasting the best-practice line

Using the advantage of the striking linear trend observed in the best-practice line, especially for females, and assuming the persistence of the observed past trends, we extrapolate the stochastic process using the classic ARIMA models. The model fitting the data best is an ARIMA(2,1,1) for females and ARIMA(1,1,1) for males. The estimated future values of the record life expectancy in the year 2050, together with the 80% and 95% prediction intervals are provided in Figure 2. The colored lines represents the extrapolation of the regression lines evaluated on the data period from 1900 to 2003.

Recalling that the future values of life expectancy at the national level can be obtained by combining the future values of the best-practice line with the future values of the gap in order to consider also the uncertainty associated with the forecasts, we have to proceed through simulation. The same number of future paths is generated both for the record life expectancy and for the gap, and then they are combined together. Simulation of the best-practice line are performed through the selected ARIMA models, randomly generating the innovation ϵ_t from a normal distribution with mean zero and variance σ_ϵ^2.

Table 1: Estimated parameters of the geometric Brownian motion applied to Italian and US, on the period 1900-2003, by sex.

	Females		Males	
	Italy	USA	Italy	USA
μ	-0.0107	0.0046	-0.0143	-0.0025
σ	0.1553	0.1569	0.1515	0.1274

The parameters of the models and the value of σ_ϵ^2 are the one returned from the fitting of the ARIMA model to the data. In mathematical terms these models are described as follows:

$$\begin{aligned}\nabla e_0^{Best}(t) &= \delta + \phi_2 \nabla e_0^{Best}(t-2) + \phi_1 \nabla e_0^{Best}(t-1) + \epsilon_t + \theta_1 \epsilon_{t-1} \\ \nabla e_0^{Best}(t) &= \delta + \phi_1 \nabla e_0^{Best}(t-1) + \epsilon_t + \theta_1 \epsilon_{t-1}\end{aligned}$$

the first equation refers to females and the second to males.

4.2 Forecasting the gap with the geometric Brownian motion

With regard to our data, the variable X presented in equation (1) represents the gap between the country specific life expectancy and the best-practice level. The model want to fit the behavior of the gap, that shows a slow increase when a country is close to the best-practice level, and a faster increase while is further away. The estimated parameters of the model μ and σ are presented in Table 1.

To obtain the distribution of the future values of the national life expectancy until the year 2050, is necessary to simulate the future values of the gap and combine them with the same number of simulated future values of the best-practice line. Here we first simulate the future values of the logarithm of the gap according to equation (5) and then apply the exponential function.

The distribution of the future values of the gaps are plotted in Figure 3 while future life expectancies are plotted in Figure 4. Satisfactory results are produced for Italy, while the US data return very unreasonable values of the prediction intervals. These results question the capability of the model to reproduce the uncertainty in the gap.

Figure 3: Forecast of the gap until the year 2050, based on the geometric Brownian motion for Italy and the US, with the corresponding 80% and 95% prediction intervals. Female (left panel) and male (right panel) data.

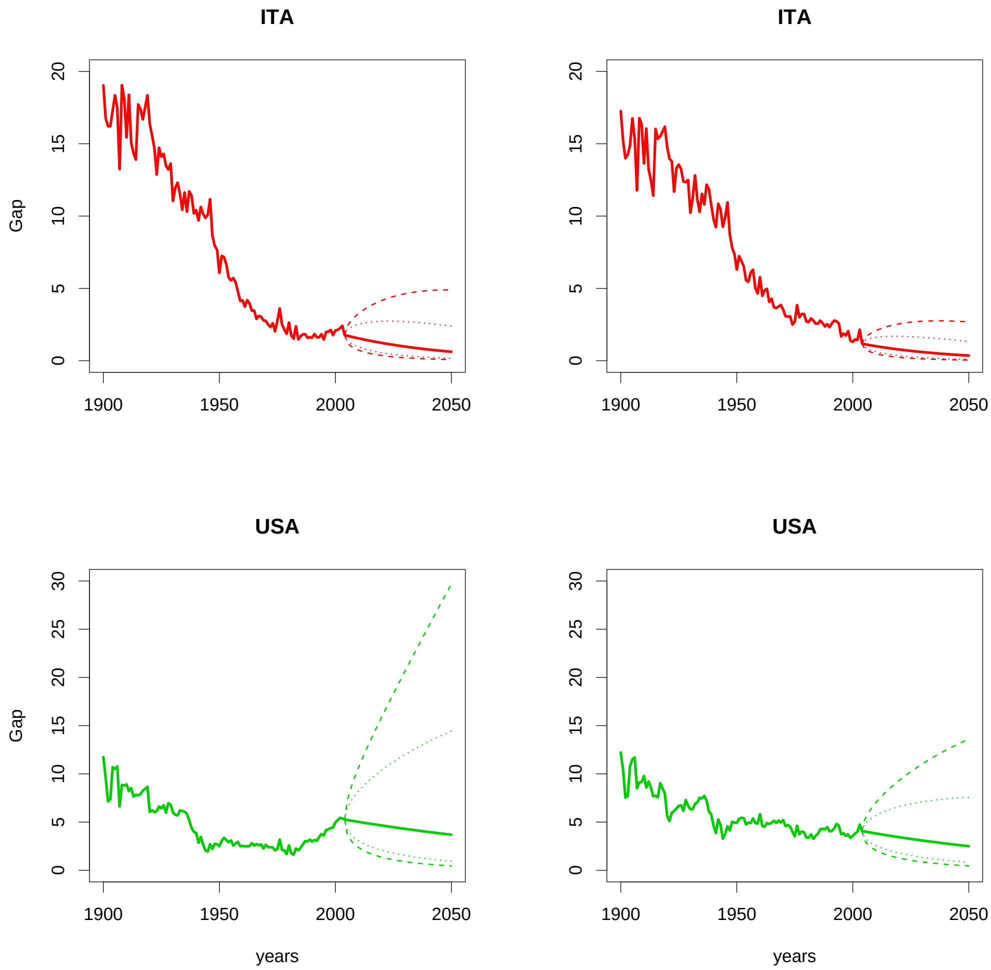
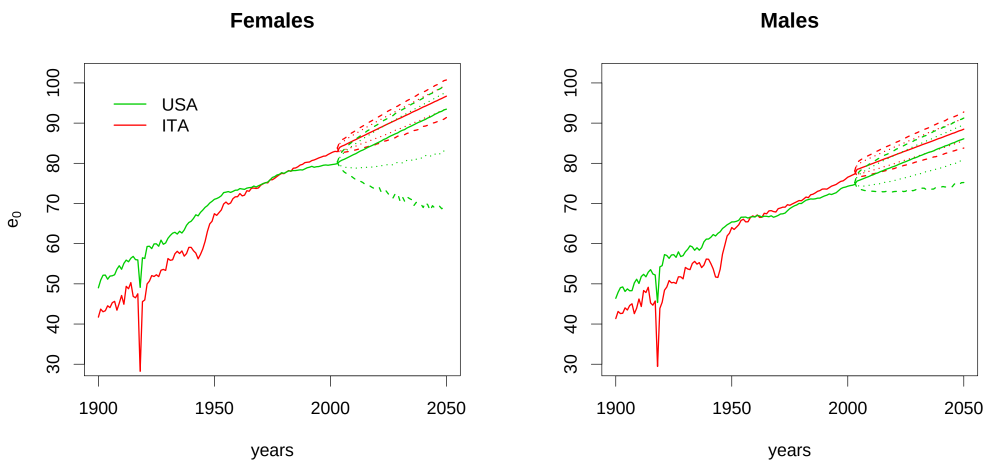


Figure 4: Forecast life expectancy until the yeas 2050, based on a geometric Brownian motion for Italy and the US, with the corresponding 80% and 95% prediction intervals.



Life expectancy for the two countries show quite a parallel trend, with the US maintaining its unfavorable position. Italian females are expected to reach in 2050 a value of life expectancy equal to 96.7. The corresponding value for males is equal to 88.4. The US are expected to have a life expectancy of 93.5 years for females and 86.1 for males.

More information is provided by the prediction intervals of life expectancy, although extremely wide and unreasonable values are obtained in the US data. The 95% prediction intervals around the Italian life expectancy in the year 2050 is 8 years wide for females and 9 years for males. As already anticipated non meaningful values of the prediction intervals for the US were obtained.

4.3 Forecasting the gap with the geometric mean reversion process

The results of the previous section, obtained applying the geometric Brownian motion to the gaps, did not show satisfactory results in terms of prediction intervals. We apply here an alternative model characterized by a non-explosive behavior that tends to fluctuate around the reversion level. Let recall the geometric mean reversion process in equation (6), describing the behavior of the gap between the country specific life expectancy and the

Table 2: Estimated parameters of the geometric mean reversion process, applied to Italian and US data, on the period 1900-2003, by sex.

	Female		Male	
	Italy	USA	Italy	USA
μ'	0.7063	1.2754	-2.3012	1.5880
η	0.0157	0.0668	0.0062	0.1081
σ	0.1559	0.1584	0.1519	0.1295

best-practice level. The estimated parameters of the model μ', η and σ are presented in Table 2.

To obtain the distribution of the future values of the national life expectancy until the year 2050, is necessary to simulate the future values of the gap and combine them with the same number of simulated future values of the best-practice line. Here we first simulate the future values of the logarithm of the gap according to equation (9) and then apply the exponential function.

The distribution of the future gaps is plotted in Figure 5 while future life expectancies are plotted in Figure 6. The estimated long-run equilibrium level seems to be very close to the last observed value of the gap. The future values of life expectancy at birth, plotted in Figure 6, show a slightly converging trend of the female life expectancies, slightly diverging for male. Italian females are expected to reach in 2050 a value of life expectancy equal to 95.7. The corresponding value for males is equal to 88.1. The US are expected to have a life expectancy of 93.8 years for females and 84.0 for males.

More information is provided by the prediction intervals of life expectancy, showing a limited variance of the process in the long-run, stable on certain values. The 95% prediction intervals around female life expectancy in the year 2050 is 11 years wide for both countries. Life expectancy for Italian males has 95% prediction intervals 9 years wide. The same value for the US is equal to 10 years.

5 Conclusions

Combining the two distinct forecasts of the best-practice line and the gap, we obtained forecasts of life expectancy. The gap was forecast with two different models, that produce quite similar results of the median value of

Figure 5: Forecast gap until the year 2050, based on a geometric mean reversion process for Italy and the US, with the corresponding 80% and 95% prediction intervals. Female (left panel) and male (right panel) data.

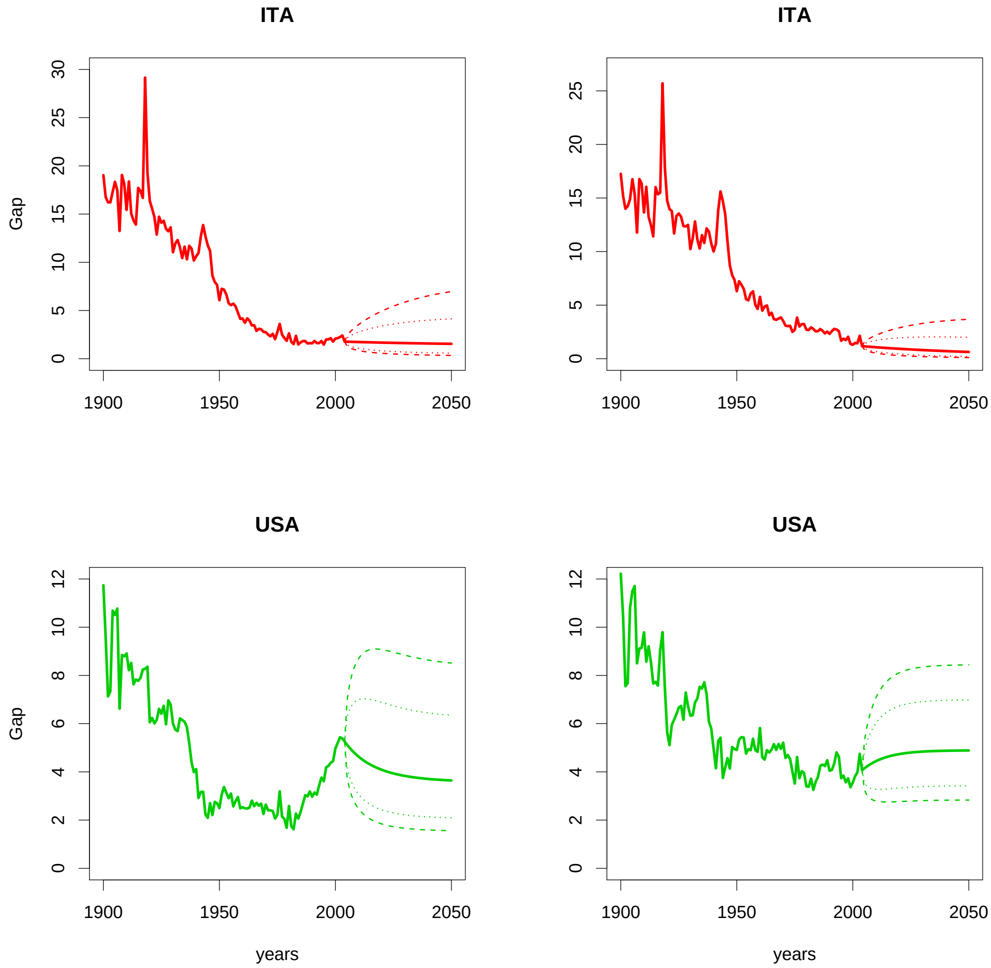
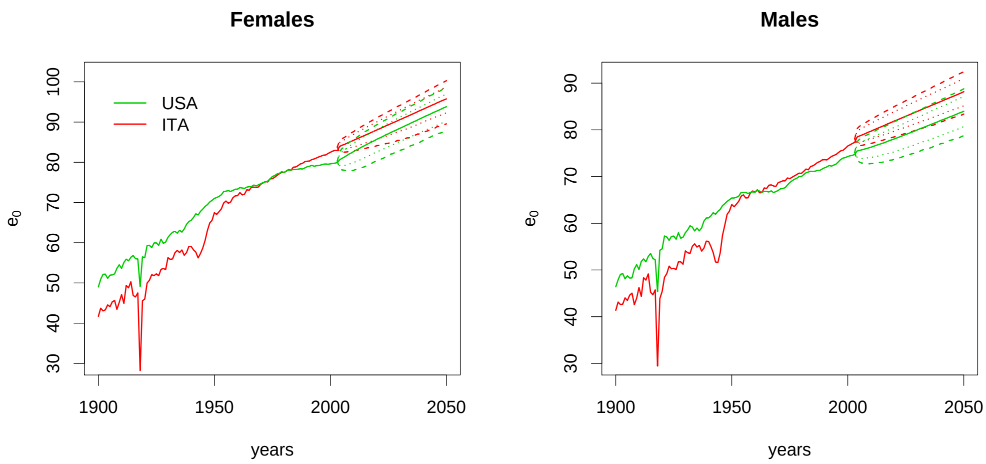


Figure 6: Forecast life expectancy based on a geometric mean reversion process for Italy and the US, with the corresponding 80% and 95% prediction intervals. Female and male data.



life expectancy in the year 2050. However, the two models differ in the estimation of the uncertainty of the forecast. The geometric Brownian motion returned sometimes explosive and unreasonable prediction intervals. More conservative, but more reasonable, are the prediction interval returned by the geometric mean reversion process. On average they returned 10 years wide prediction intervals.

Let stress here which is the strength of the models presented. It occur, sometimes, forecasting life expectancy through age-specific death rates or directly forecasting life expectancy, to obtain extremely high or low future values. You can see plots of future life expectancy bending over time or, on the contrary, increasing extremely fast. This happen especially if forecasts are performed considering data from the beginning of the century (1900-2004). Moreover, these results are not consistent with the acknowledged long term trend in the best-practice line. Working within the framework of the best-practice line we have been able to constrain future life expectancy within reasonable values, coherently with the trend of the best-practice line.

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