Life Expectancy and Disparity

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We discuss the remarkable correlation between life expectancy and life disparity across countries and over time. We define life disparity as $e^{\dagger} = \int_0^{\omega} e(x,t) f(x,t)dx$,

where $e(a,t) = \dfrac{\int_a^{\omega} l(x,t)dx}{l(a,t)}$ is remaining life expectancy at age *a* and time *t,*

$l(a,t) = \exp\left(-\int_0^a \mu(x,t)dx\right)$ gives the probability of survival to age *a* and $\mu(a,t)$ denotes the age-specific hazard of death. The life table distribution of deaths is given by $f(a,t) = l(a,t)\mu(a,t)$. Maximum lifespan is denoted by *ω.*

Saving a life at any age extends life expectancy. Saving a life at an old enough age increases life disparity; saving a life at a young enough age decreases life disparity. We prove that if a threshold age between such late and early deaths exists, then this age is unique. The proof is as follows:

Consider the increase in $e^{\dagger}$ due to reductions in mortality,

(1) $$g(a,t) = \frac{de^{\dagger}(t)}{-d\mu(a,t)/\mu(a,t)} = f(a,t)k(a,t),$$

where

(2) $$k(a,t) = e^{\dagger}(a,t) - e(a,t)(1 - H(a,t)),$$

where $H(a,t) = \int\limits_0^a \mu(x,t)dx$ is the cumulative hazard function and

$e^{\dagger}(a,t) = \dfrac{\int_a^{\omega} e(x,t)f(x,t)dx}{l(a,t)}$ is life expectancy lost due to death among people surviving to age *a*. The function $g(a,t)$ measures how much $e^{\dagger}$ will be increased by a proportional reduction in mortality at age *a* and time *t*. Because *f* is always positive, if *k* is negative, then the change decreases life disparity; if *k* is positive, then the change increases life disparity. If *k* is negative at younger ages and positive at older ages, then there is some age $a^{\dagger}$ at which *k* equals zero. This is the age that separates early deaths from late deaths. We prove below that $a^{\dagger}$ exists under conditions that generally characterize modern human populations. Furthermore we prove that if $a^{\dagger}$ exists then there is one and only one age, $a^{\dagger}$, at which *k* equals zero.

For notational simplicity, the time subscript *t* can be dropped without confusion. Let

(3) $$k(a) = e^{\dagger}(a) - e(a)(1 - H(a)),$$

so that

(4) $$k(0) = e^{\dagger}(0) - e(0).$$

We consider three cases.

Case 1: $k(0) < 0$.

At advanced ages, as $a \to \omega$, both $e^{\dagger}(a)$ and $e(a)$ approach 0, but $H(a)$ approaches $+\infty$, and thus $k(\omega) > 0$. The function $k(a)$ is continuous on $[0,\omega]$. According to the intermediate value theorem, there exists at least one point, say $a^{\dagger}$, in $[0,\omega]$ such that $k(a^{\dagger}) = 0$.

It is readily shown that there is only one $a^{\dagger}$ in $[0,\omega]$ such that $k(a^{\dagger}) = 0$. If there were more than one point at which $k(a)$ equals zero, then the derivative of $k(a)$ at some of these points would be positive and at others negative, because the continuous function $k(a)$ must go up and down to cross zero more than once. If the derivative of $k(a)$ is always positive when $k(a) = 0$, then there is only one point at which $k(a)$ crosses zero.

The derivative of (Eq. 3) is given by

$$\frac{dk(a)}{da} = \frac{de^{\dagger}(a)}{da} - (1-H(a))\frac{de(a)}{da} - e(a)\frac{d(1-H(a))}{da}$$

$$= -\mu(a)e(a) + \mu(a)e^{\dagger}(a) - (1-H(a))(-1+\mu(a)e(a)) + e(a)\mu(a)$$

(5) $$= \mu(a)e^{\dagger}(a) + (1-H(a))(1-\mu(a)e(a)).$$

When $k(a) = 0$ it follows from (3) that $1-H(a) = \frac{e^{\dagger}(a)}{e(a)}$. Substituting this into (5) yields

(6) $$\left.\frac{dk(a)}{da}\right|_{a=a^{\dagger}} = \mu(a)e^{\dagger}(a) + \frac{e^{\dagger}(a)}{e(a)}(1-\mu(a)e(a)) = \frac{e^{\dagger}(a)}{e(a)} > 0.$$

Case 2: $k(0) = 0$.

In this case, $a^{\dagger} = 0$. We need to show that this is the only value of $a^{\dagger}$, i.e., the only age when k=0. It follows from (5) that if $k(0) = 0$ then

$$\left.\frac{dk(a)}{da}\right|_{a=0} = 1.$$

Hence $k(a)$ becomes positive as age increases from zero. If there were an age above zero when $k(a) = 0$, then the derivative of k at this age would have to be zero or negative. But as shown in (6), the derivative has to be positive at any age when $k(a) = 0$. This contradiction implies that the value of $a^{\dagger} = 0$ is unique in the case when $k(0) = 0$.

Case 3: $k(0) > 0$.

As noted above in Case 2, if there were an age when $k(a) = 0$ then the derivative of k at this age would have to be zero or negative. But as shown in (6), the derivative has to be positive at any age when $k(a) = 0$. This contradiction implies that there is no age that separates early from late deaths when $k(0) > 0$: averting a death at any age would increase life disparity. Hence in this case it is convenient to set $a^{\dagger}$ equal to zero by definition. Q.E.D.

We have computed the value of $k(0)$ for all 5830 life tables since 1840 in the Human Mortality Database (2008), the life tables used in this article. We have also computed the value of $k(0)$ for the 3404 life tables in the Human Life-Table Database (2008). In every case $k(0) < 0$. The closest approach to zero was found for females in

1911-1921 in India: for this population $e(0) = 23.33$, $e^{\dagger} = 23.08$, so that $k(0) = -0.22$. Goldman and Lord (1986), however, provide two examples of life tables for which $k(0)$ is positive. Both pertain to selected populations in rural areas of China in the period 1929-31. One is for females (Barclay et al., 1976) and the other is for males (Coale and Demeny, 1983). For the Chinese women $e(0)$=21.00 and $e^{\dagger}$=21.73. For the Chinese males, $e(0)$=17.43 and $e^{\dagger}$=22.17.

The entropy of life tables (Keyfitz, 1977) is equal to $e^{\dagger}/e(0)$. In addition to $e^{\dagger}$, several other measures of the life disparity in a lifetable have been proposed (Cheung et al., 2005). These include the variance in the age at death, the standard deviation, the standard deviation above age 10 (Edwards and Tuljapurkar, 2005), the inter-quartile range (Wilmoth and Horiuchi, 1999), and the Gini coefficient (Shkolnikov et al., 2003). These measures are highly correlated with each other. According to authors' calculation based on the period life tables available at Human Mortality Database, the correlation of $e^{\dagger}$ with the other measures never falls below 0.964 for females and 0.933 for males. Hence $e^{\dagger}$ can be viewed as a surrogate for the other measures. We prefer $e^{\dagger}$ because of its desirable mathematical properties and because it can be readily explained and interpreted.

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