THE IMPACT OF DISPARITIES ON THE AGE TRAJECTORY OF MORTALITY: A FRAILTY MODEL USING TRUNCATED NORMAL DISTRIBUTIONS

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This paper proposes a new approach to modeling mortality disparities in heterogeneous populations. First, we define a gamma-Gompertz mortality trajectory for a heterogeneous "best-practice" group (BPG). The BPG trajectory represents the lowest attainable mortality given optimal scientific knowledge, health behaviors, and access to medical treatment and technology at any given point in time. Next, we derive expressions relating the mortality experience of any population to the BPG, a relationship we define as a new model of mortality disparities. Maximum likelihood methods and data from the Human Mortality Database (HMD 2009) are then used to simultaneously estimate a key parameter of the BPG trajectory as well as parameters of the truncated normal distribution describing mortality in any population of interest relative to the BPG. This approach is an expansion of the frailty model developed by Vaupel et al. (1979). Its added flexibility allows the model to account for differences in the distribution of frailty across populations relative to a mortality frontier that improves over time.

The Best Practice Group

Individual members of any given population differ in their general susceptibility to all causes of deaths. The classic frailty model (Vaupel et al. 1979) multiplicatively defines the mortality hazard of any individual at age x and frailty level z with respect to the mortality hazard of a standard individual or sub-group:

$$\mu(x,z) = z\mu(x), \tag{1}$$

where z is gamma-distributed with mean one and variance σ^2.

We now consider a modification of the frailty model that uses as reference a best-practice group (BPG) with the lowest-attainable mortality hazard rather than a theoretical standard. The BPG's trajectory represents the lowest attainable mortality, given the most advanced knowledge, leading technology, and best medical treatment available at the time, by a group of individuals with ideal health habits living in a place that fosters the highest life expectancy. The hazard function for a (standard) member of the BPG increases exponentially with age x, following a Gompert curve, $\mu^*(x) = \alpha e^{\beta x}$. The BPG group is heterogeneous in its members' susceptibility to mortality, with some individuals in this group frailer than others, perhaps for genetic reasons or due to early-life experiences. This internal heterogeneity results in a bending BPG mortality curve that may be described using the logistic form of the Gamma-Gompertz curve:

$$\bar{\mu}^*(x) = \frac{\alpha e^{\beta x}}{1 + \frac{\sigma_z^2 \alpha}{\beta}(e^{\beta x} - 1)}. \quad (2)$$

Below, we discuss the estimation procedure for the key parameter α, representing a background level of mortality that may change over time. The two other parameters are set at $\beta = 0.13$ and $\sigma^2 = 0.2$, consistent with findings in previous research on long-lived populations.

The Disparity Model

Consider the following disparity model:

$$\bar{\mu}(x, \delta_i) = (1 + \delta_i)\bar{\mu}^*(x), \quad (3)$$

where δ_i represents the gap or disparity between the age-specific mortality hazard in the best-practice group (BPG), $\bar{\mu}^*(x)$, and the age-specific mortality hazard $\bar{\mu}(x, \delta_i)$ for any whole population.

Let $f_\delta(\delta)$ denote the marginal probability distribution function *(p.d.f)* of δ. Then, the age-specific force of mortality for any individual in a population with disparity δ may be defined by:

$$\mu(x,\delta) = \frac{f_{x|\delta}(x \mid \delta)}{s(x,\delta)}, \tag{4}$$

where $s(x,\delta)$ is the survival function is defined by:

$$s(x,\delta) = \exp\left(-\int_0^x \mu(t,\delta)dt\right), \tag{5}$$

and $f_{x|\delta}(x \mid \delta)$ is the conditional *p.d.f* of x given δ:

$$f_{x|\delta}(x \mid \delta) = \mu(x,\delta)s(x,\delta). \tag{6}$$

The joint *p.d.f* of x and δ is thus:

$$f_{x,\delta}(x,\delta) = f_{x|\delta}(x \mid \delta) \cdot f_\delta(\delta). \tag{7}$$

The average mortality observed for the population i at age x is essentially the expected value[1] with respect to δ:

$$\bar{\mu}_i(x) = \frac{\int_0^\infty \mu(x,\delta)f(x,\delta)d\delta}{\int_0^\infty f(x,\delta)d\delta}. \tag{8}$$

[1] The variable δ represents the degree of heterogeneity in frailty in the population of interest, relative to the BPG. While it may theoretically range from negative infinity to positive infinity, the nature of frailty (defined as a proportionate hazard relative to a standard trajectory) constrains delta to the positive realm, with the integral left-truncated at 0.

The Distribution of δ

The distribution of $f_{\delta}(\delta)$ describes both the average gap between the population of interest and the BPG (through the distribution's mean, $\overline{\delta}$) as well as the internal heterogeneity of the population (through the distribution's variance). Notably, in the classic frailty model heterogeneity was defined relative to the mortality hazard of a standard individual or subgroup, and other members of the population could be either more or less frail than the standard. In the disparity model, however, the heterogeneity is defined with respect to the BPG, which has the lowest attainable mortality hazard at any given time. A given population *i* therefore consists of individuals whose hazard trajectory is either equal to or higher than the BPG trajectory, with the exact distribution defined by the parameters of δ_i.

If δ is gamma-distributed, equations (6)-(8) are tractable (see Vaupel et al. 1979). However, the gamma distribution is not fully suitable for the disparity model. As shown in Figure 1, the frailty described by the disparity parameter δ in equation (3) may be distributed in three different shapes . In a type 1 scenario, a large proportion of the population may have a value of δ equal to or close to 0, rendering their mortality trajectory essentially equal to that of the BPG. This distribution is also, however, characterized by a large variance, with a substantial proportion of the population experiencing higher frailty leading to an overall high level of disparity in the population. Under a type 2 scenario, the frailty curve resembles that of a normal distribution, with only a slight proportion of the population experiencing the BPG trajectory (since $f(0)$ is nearly 0). In this scenario, however, the variance of δ is smaller than the variance under a type 1 scenario, and thus the distribution of mortality within the population is more equal. Finally, a type 3 scenario describes a population where a certain proportion of the population experiences the BPG mortality trajectory

(more than under scenario 2, but less than scenario 1) and where the variance is greater than it is under scenario 3 but less than under scenario 1.

Notably, the gamma distribution can reflect scenarios 1 and 2, but not 3. However, as will be shown below, scenario 3 is often observed in empirical data. The truncated normal distribution provides a more flexible model that can capture the features of all three scenarios discussed above and is consequently used to describe δ in the remainder of this paper. Thus, we assume $\delta \sim N(\bar{\delta},\sigma^2), \delta \in [0,b]$, with *p.d.f*

$$f(\delta) = \frac{\frac{1}{\sigma}\phi\left(\frac{\delta-\bar{\delta}}{\sigma}\right)}{\Phi\left(\frac{b-\bar{\delta}}{\sigma}\right)-\Phi\left(\frac{-\bar{\delta}}{\sigma}\right)}, \tag{9}$$

where $\phi(\cdot)$ is the *p.d.f* of the standard normal distribution and $\Phi(\cdot)$ is its cumulative distribution function.

Numerical integration may be used to evaluate the expected value in equation (8). A comparison of the numerical integration approach with the closed-form results under the Gamma distribution yielded nearly identical results, confirming the appropriateness of numerical integration as a method for obtaining an estimate of $\bar{\mu}(x)$ under the assumption that δ follows a truncated normal distribution.

Note that any population of interest may thus include some proportion of people with the BPG hazard trajectory, as well as subgroups whose hazard trajectory is higher than that of the BPG. The parameters of the truncated normal distribution $(\bar{\delta},\sigma^2)$ for each population of interest uniquely characterize each population's mixture of mortality trajectories. At the same time, the common reference to the BPG facilitates cross-national comparisons by accounting for differential heterogeneity in mortality.

Maximum Likelihood Estimation

A closed form likelihood function was constructed to estimate the parameters of the disparity model. For a given life table cohort (real or synthetic), we denote the number of survivors to exact age x by l_x, and the number of deaths between exact ages x and $x+1$ as d_x. Then, the probability of dying at age x is $q_x = d_x / l_x$. Given $\bar{\mu}(x)$ as in equation (8), we can approximate the probability of death at age x by

$$q_x \cong 1 - e^{-\bar{\mu}(x+0.5)}. \tag{10}$$

This probability may be considered as the underlying parameter in l_x Bernoulli trials of which d_x are failures and $l_x - d_x$ are successes. Using the approximation in equation (10), the likelihood function for the parameter q_x representing the probability of death at age x given the observed number of deaths may be characterized using the binomial distribution:

$$L(q,x) = \binom{l_x}{d_x} q_x^{d_x} (1-q_x)^{l_x - d_x} \tag{11}$$

Given life table data on n populations, where $\mathbf{X}^{(i)} = (d_x^{(i)}, l_x^{(i)})$ is the vector of observed data for the ith population, the joint likelihood can be used to estimate the set of parameters of interest, θ:

$$L(\theta \mid \mathbf{X}) = \prod_{i=1}^{n} L_i(\alpha, \bar{\delta}_i, \sigma_i^2 \mid d_x^{(i)}, l_x^{(i)}). \tag{12}$$

α represents the Gompertz parameter defining the BPG across all populations (equation 3) while $\bar{\delta}_i$ and σ_i^2, $i = 1,2,...,n$ are the parameters of the disparity distribution for the ith population. All

parameters are estimated simultaneously given the observed data for all n populations, and the maximum likelihood estimate (MLE) of θ given $\mathbf{X}$ is thus

$$\hat{\theta} = \arg\max_{\theta} L(\theta \mid \mathbf{X}) \tag{13}$$

Data

The Human Mortality Database (HMD, 2009) contains detailed time series of mortality data and life tables for populations with virtually complete registration and census data. Data on females from 21 countries with reliable life tables going back to 1950 were included in the analysis: Australia, Austria, Belgium, Canada, Czech Republic, Denmark, England & Wales, Finland, France, Hungary, Iceland, Ireland, Italy, Japan, the Netherlands, New Zealand, Norway, Portugal, Sweden, Switzerland and the United States. Five-year period life tables were used to improve the smoothness of the estimated functions.

Results

The α parameter represents baseline mortality in the Gompertzian BPG model. The left panel of Figure 2 shows that the value of α has declined substantially over the past half century. The right panel shows that the declining parameter corresponds to an age-trajectory of mortality hazard that has shifted steadily downward over time, consistent with previous findings regarding the logistic trajectory of adult mortality (Bongaarts 2005). The trajectories for 1950-4, 1970-4, and 2000-4 are highlighted in red, violet, and blue, respectively, to show progress over time. Notably, while the two early curves show a smaller improvement in mortality at older ages relative to mortality at younger ages, the difference between the 1970-4 and 2000-4 curve shows clear mortality reductions even at

the oldest ages in recent times. This result is in line with research showing steady improvements in survival in record-holding national populations (Oeppen and Vaupel 2002) and among the oldest-old (Rau et al. 2008). Even more than record-holding countries, the subpopulations comprising the BPG represent a frontier in survival, showing the effect of improvements in knowledge, technology, and access to health-promoting resources on the pattern of mortality over time.

Next we turn to the parameter estimates for national populations. Figure 3 shows trends in the estimated means of the truncated normal distributions for four selected countries and Figure 4 displays trends in the variance of the disparity distributions for the same countries. The mean for England and Wales changes only slightly between 1950-1970, but increases notably after 1975, staying at an elevated level until 1995, when a decline is noticeable. In contrast, the mean disparity in France shows a general pattern of decline over time with some fluctuation until 1995, when a noticeable increase is apparent. The U.S. trajectory shows a declining mean disparity between 1955 and 1970, and then, starting in 1980, a notable increase that leaves the U.S. with the highest mean disparity out of the four nations in 2000-4. Japan offers the most marked contrast: starting at a very high mean disparity in 1950, it declines consistently over time, and achieves the lowest mean disparity among these four countries after 1980, a trajectories that is consistent with Japan's rapidly rising life expectancy during the same period. Like the other three nations, it also shows an increase in mean disparity in the most recent period.

The variance of the disparity distribution in the four countries follows a pattern closely resembling that of the means (see Figure 4). In Figure 5, calculated means and variances for all 21 countries are plotted for every 5-year period between 1950-2004. The correlation between mean and variance in the disparity model is clearly positive: more variance in the distribution of disparity suggests a higher proportion of the population outside the BPG, and thus a higher mean disparity.

The four plots in Figure 6 display the distributions of disparity relative to the BPG in each of

the four countries. Each curve is plotted based on the given year's α value and the year-and-country-specific estimates of the truncated normal parameters. The disparity distribution in all four countries may be classified as "type 3," though the exact parameters vary, reflecting differences in population heterogeneity across countries. The curves for the years 1950-4, 1970-4, and 2000-4 are highlighted in red, violet, and blue, respectively. In England and Wales, the distribution of disparity is relatively unchanged between 1950-4 and 197004, and becomes flatter by 2000-4, suggesting somewhat greater inequality relative to the BPG than in the past. In France, the distribution becomes more peaked (i.e. more equal) between 1950-4 and 1970-4, but then flattens out somewhat by 2000-4. The gray lines suggest distributions with lower means and variances (i.e. more equal distributions) during some of the non-highlighted years. In Japan, the trend is for increasingly peaked distributions over time, though the existence of a few curves above the blue curve indicate that in some years previous to 2000-4 Japan's disparity distribution was somewhat more equal than it was in the later period. In the U.S., two contrasting trends are apparent. Between 1950-4 and 1970-4, the distribution followed the pattern observed in France and Japan, become more peaked and indicative of declining inequalities. By 2000-04, however, the distribution of disparity relative to the BPG had become flatter, showing more inequality relative to the BPG than in any previous year or any of the countries.

Figure 7 displays the hazard trajectories associated with the estimated disparity parameters. England and Wales and France show similar levels of mortality and patterns of change over time, with improvement relative to the BPG between 1950-4 and 1970-4 and then slightly poorer performance relative to the BPG in the most recent period. In Japan, progress against mortality is manifested in consistently lower hazard trajectories. The difference between the 1950-4 (red) curve and the 1970-4 (violet) curve is indicative of the mortality reductions that took place over the course of those two decades. Between 1970-4 and 2000-4 survival continued to improve, but the

magnitude of the change relative to the BPG was smaller. In the United States, there was clear improvement between 1950-4 and 1970-4, but the trajectory for 2000-4 is higher than the 1970-4 trajectory, reflecting relatively poorer performance than in the past even while mortality continues to decline in absolute levels.

Finally, Figure 8 presents three snapshots comparing the BPG hazard trajectories with those of Japan and the United States over time. The BPG trajectory, a moving comparison target, was steadily declining throughout this period. In 1950-4, the United States population as a whole was closer to the BPG trajectory than the Japanese population. By 1970-4, Japan had made considerable progress against mortality, and its hazard trajectory was just slightly higher at every age than the U.S trajectory. By 2000-4, however, Japan had surpassed the U.S., which was further than the BPG than ever before, despite ongoing improvement to survival in both countries.

Discussion

The disparity model expands the capacity of the frailty model to account for heterogeneity within and across populations. The first contribution of this model is the definition and estimation of the BPG trajectory, a moving frontier representing declining mortality under the best attainable conditions in any given time. Furthermore, the disparity model clearly expresses inequalities within populations while facilitating comparisons of mortality experiences across heterogeneous populations.

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Figure 1. Possible distributions of the disparity parameter δ

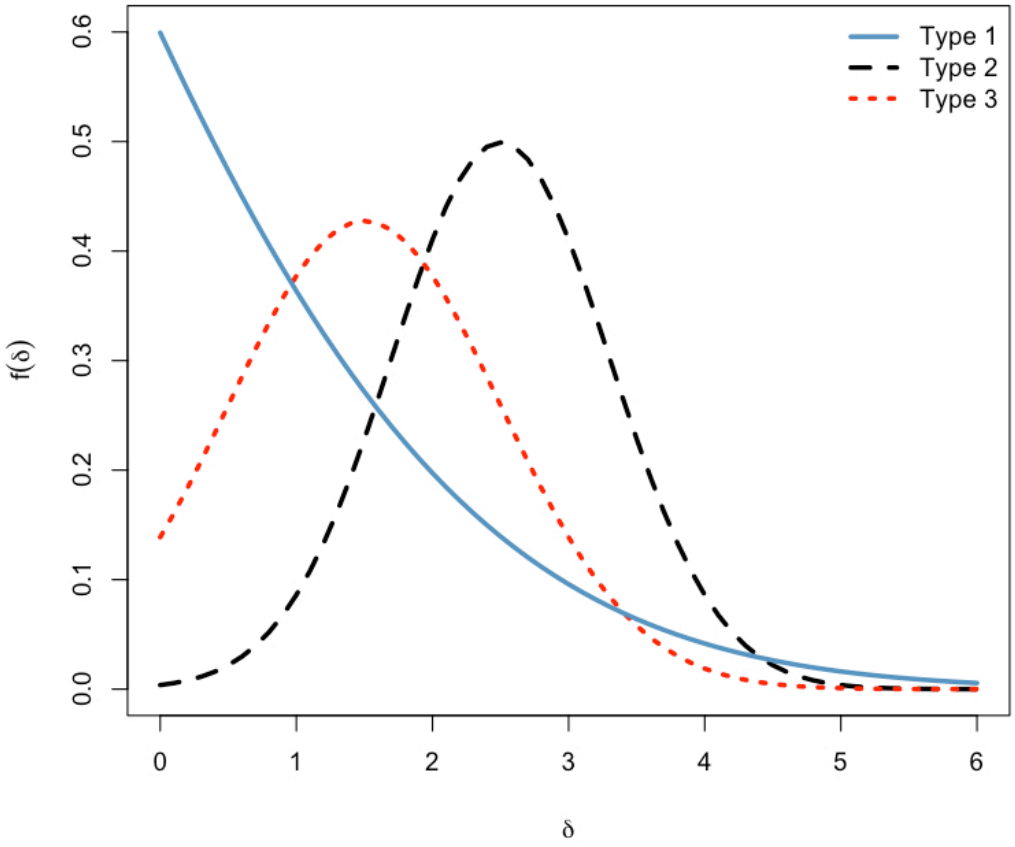

Figure 2. Declining BPG mortality: Estimates of the Gompertz parameter α and the associated (log) hazard trajectories 1950-2004.

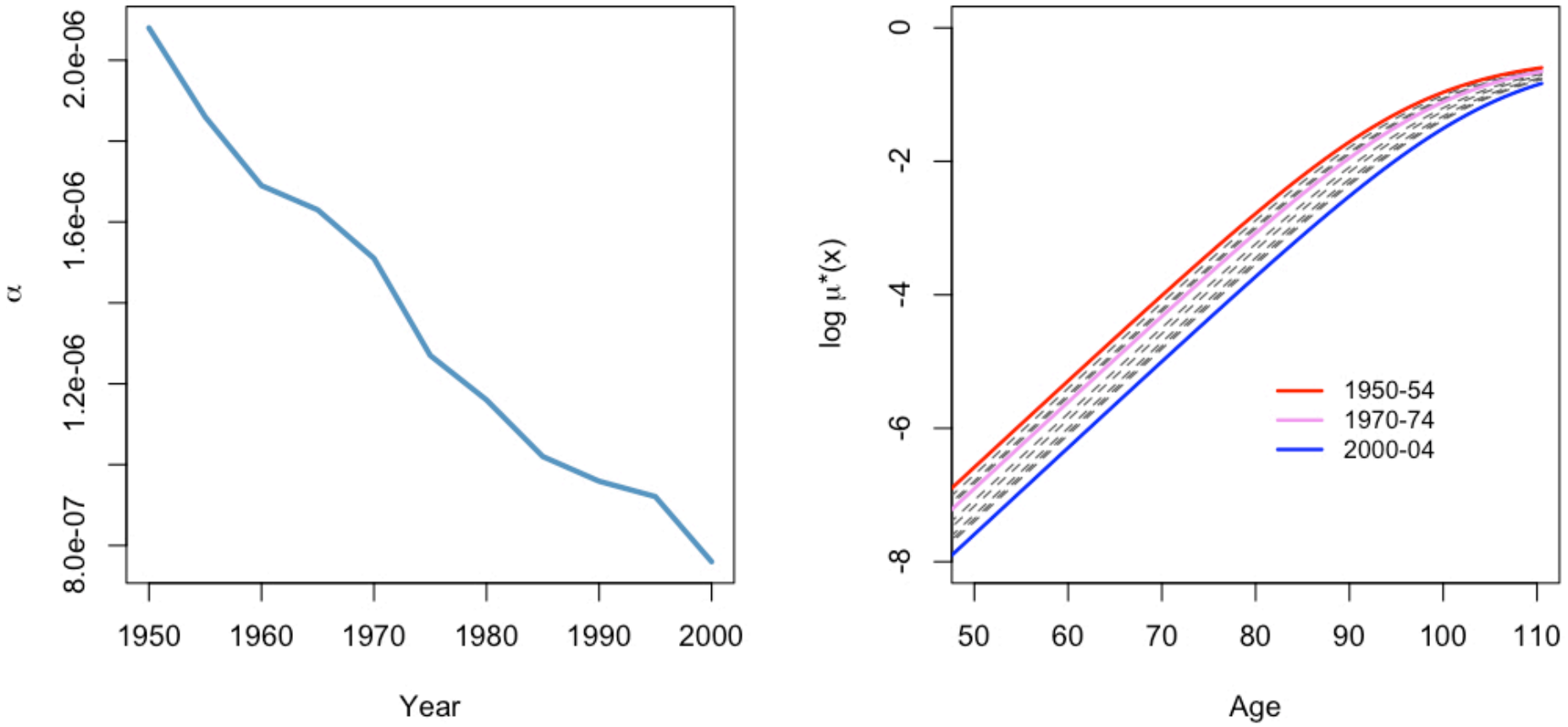

Figure 3. Estimates of mean disparity ($\overline{\delta}$) for 4 countries, 1950-2004.

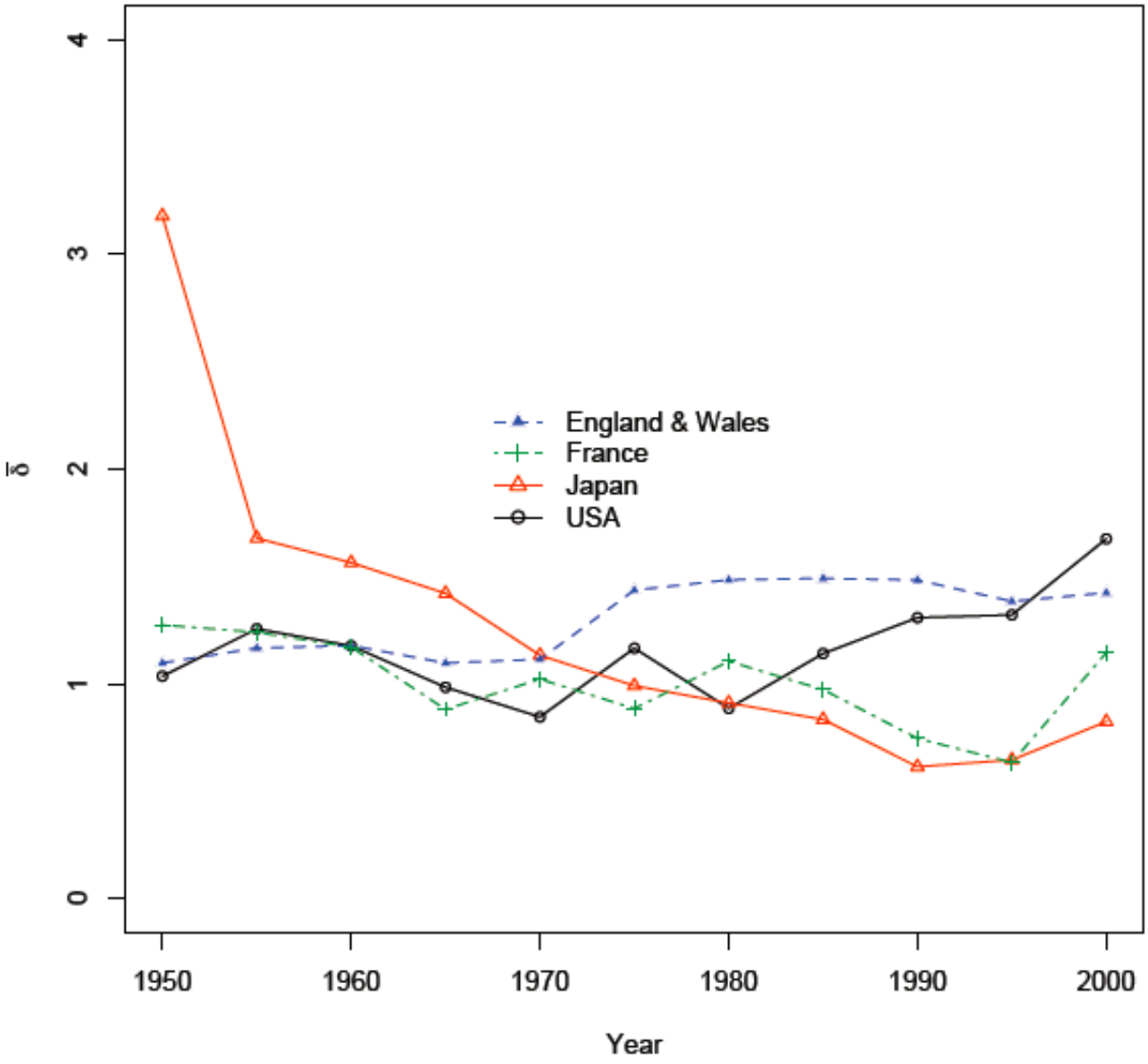

Figure 4. Estimates of the variance in the disparity distribution for 4 countries, 1950-2004.

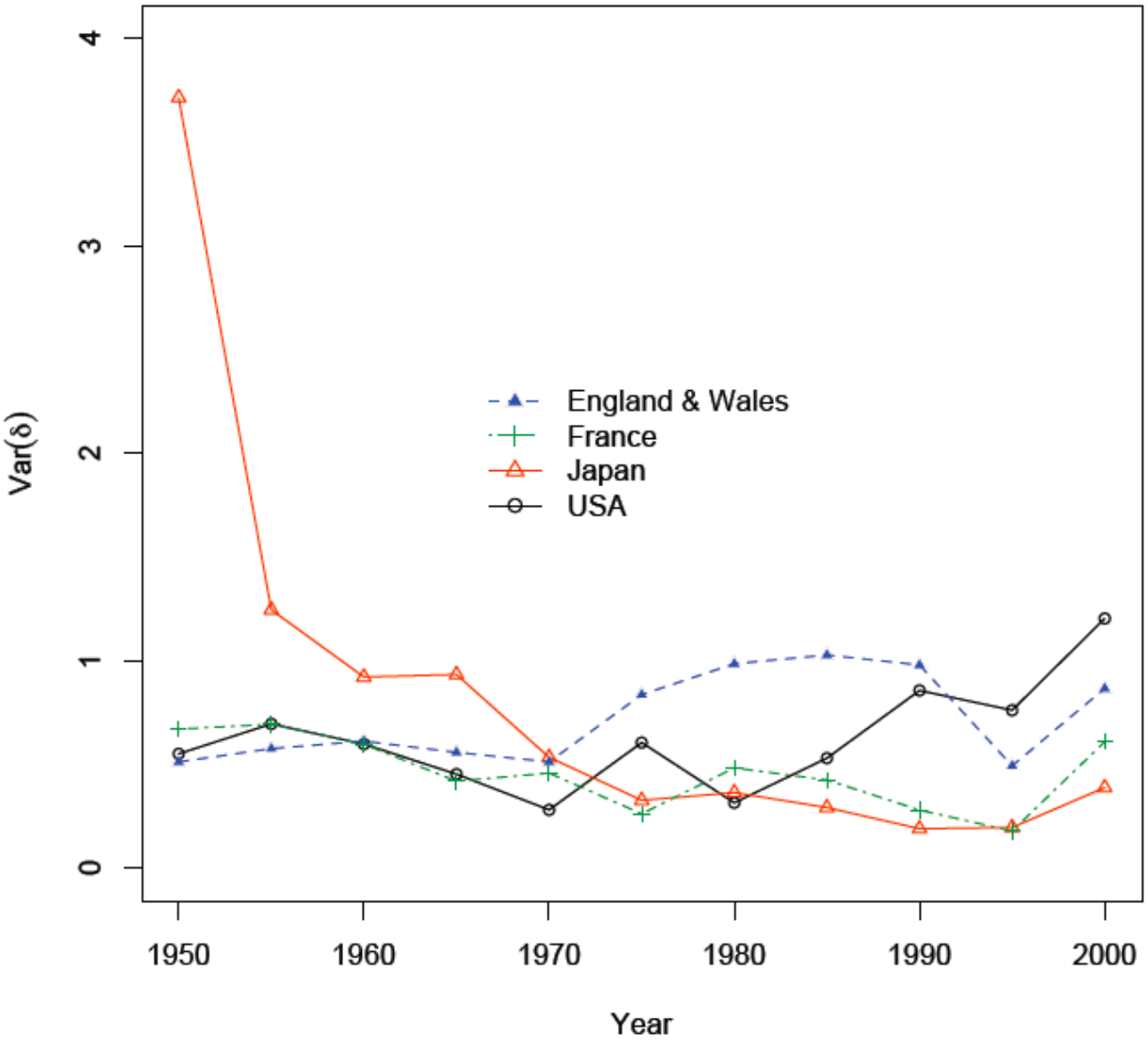

Figure 5. Mean vs. variance of the disparity distribution for 21 countries, 1950-2004.

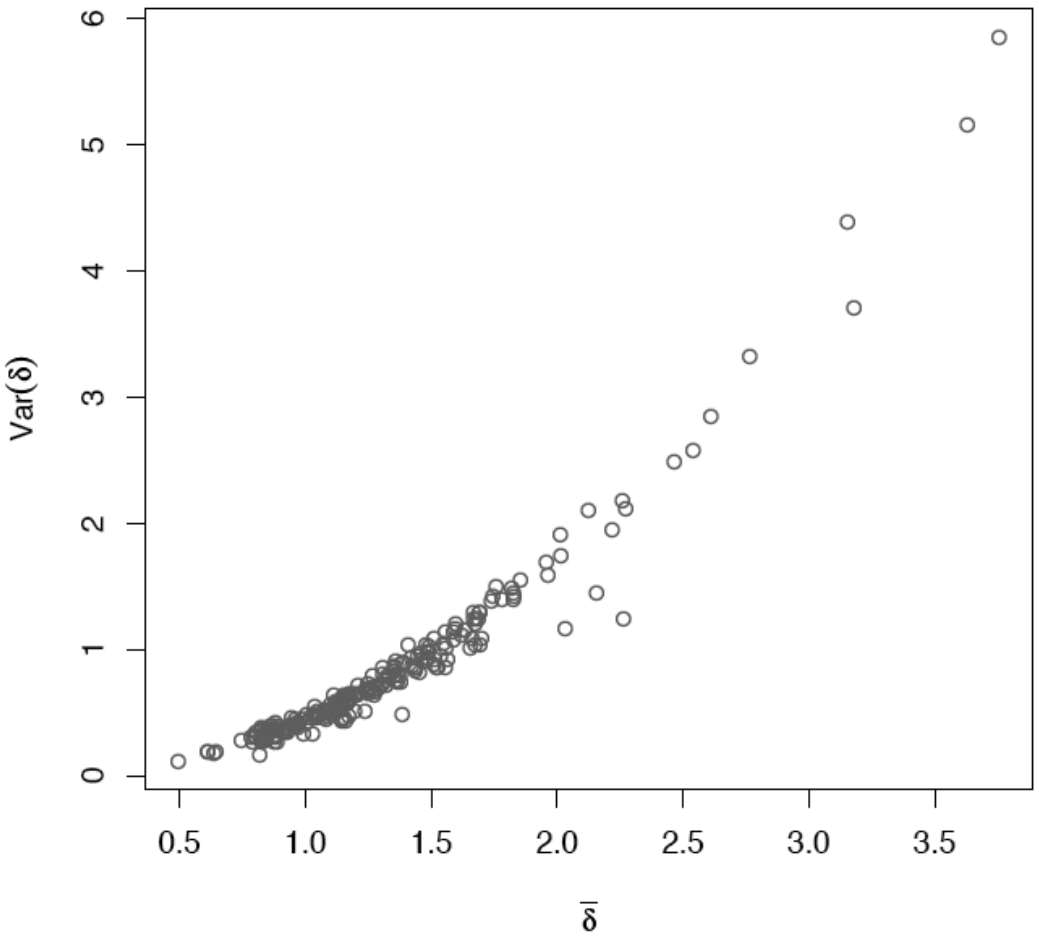

Figure 6. Disparity distributions in four countries

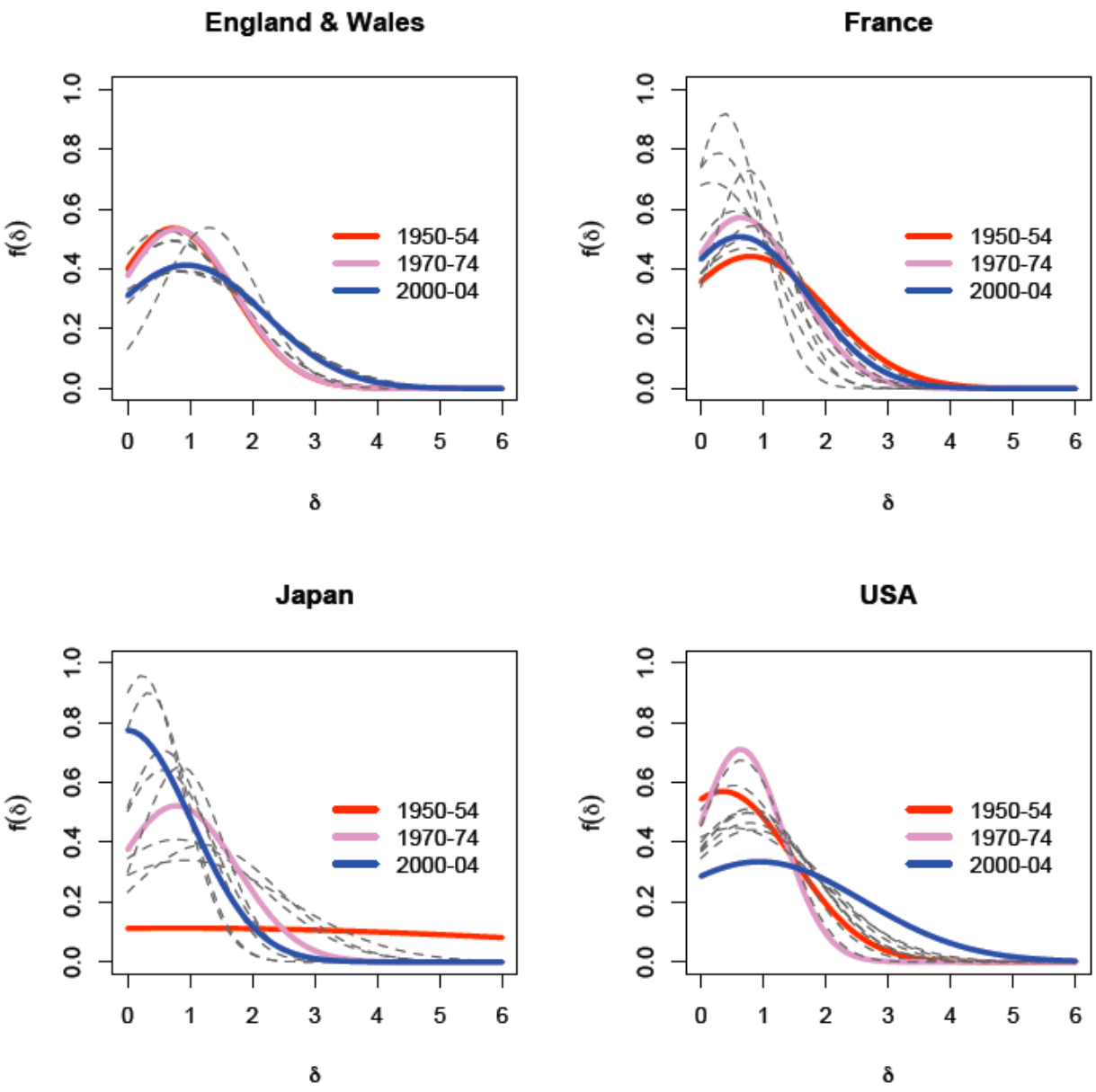

Figure 7. Associated Hazard trajectories in 4 Countries

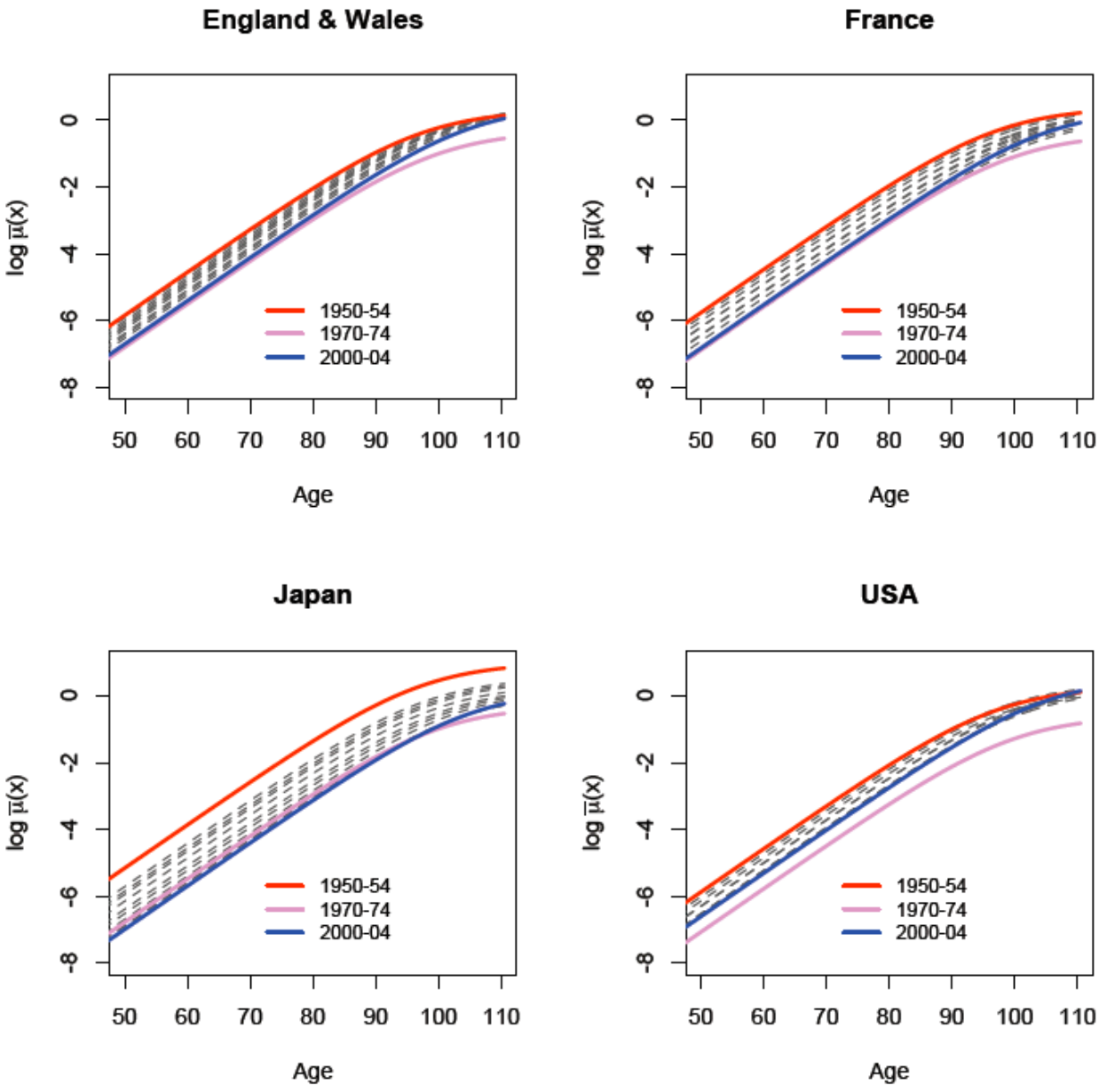

Figure 8. Hazard trajectories in two countries relative to the BPG